# Approximation Algos.

## Lecture 10

## MAX SAT

eg: $x_1 \wedge (\bar{x}_2 \vee x_3 \vee x_4)$
$\wedge (\bar{x}_1 \vee x_2) \wedge \bar{x}_2$

~~i/p~~: $n$ Bool. variables $x_1, x_2, \ldots, x_n$

$m$ clauses $C_1, C_2, \ldots, C_m$

nonnegative wt. $w_j$ for clause $C_j$, $j \in [m]$

Objective: Find an assignment to variables that maximizes sum of weights of satisfied clauses

Reading Exercise: Sections 5.1 & 5.2 in WS

$\frac{1}{2}$ - approx. to MAXSAT

Now: Better algos.

Algo:

1. Set each variable to T independently with probability $p > \frac{1}{2}$.

Assumption 1: no negated unit clauses. $\bar{x}_2$

Lemma: Algo. above is a $\min\{p, 1-p^2\}$-approx. algo for MAXSAT

Proof:

For clause $C$ :

if unit clause, satisfied w.p. $p$

otherwise it has $\geq 2$ literals.

Say, $a$ positive literals & $b$ neg. literals.

$\Rightarrow \Pr(C \text{ is satisfied})$

$= 1 - (1-p)^a \cdot p^b$ $\qquad (1-p) < p$

$> 1 - p^{a+b} \geq 1 - p^2$

$a+b \geq 2$

$= \sum_{j \in [m]} w_j \cdot \Pr[C_j \text{ is satisfied}]$

$\therefore$ Expected weight of soln. $\Big\} \geq \min\{p, 1-p^2\} \cdot \sum_{j \in [m]} w_j$

$\geq \min\{p, 1-p^2\} \cdot OPT.$

Setting $p = 1 - p^2$, we get $p = \frac{1}{2}(\sqrt{5} - 1) \approx 0.618$

## Removing Assumption 1

Assumption 2 : weight of unit clause $x_i \geq$ weight of unit clause $\bar{x}_i$ $\forall i \in [n]$

(w.l.o.g)

Defn: $v_i$ - weight of clause $\bar{x}_i$ for $i \in [n]$

if $\bar{x}_i$ does not appear as a clause, $v_i \triangleq 0$

**Lem:** $\text{OPT} \leq \sum_{j \in [m]} w_j - \sum_{i \in [n]} v_i$ $\qquad x_i \qquad \bar{x}_i$

**Lem:** Algo is a $\left(\frac{\sqrt{5}-1}{2}\right)$-approx algo for MAXSAT

**Proof:**

$U$ - indices of clauses that are not unit negated clauses

Expected wt. of solution $\geq \sum_{j \in U} w_j \cdot \Pr(C_j \text{ is satisfied})$

$\frac{\sqrt{5}-1}{2}$ $\qquad \geq p \cdot \sum_{j \in U} w_j$

$$= p \cdot \left( \sum_{j=1}^{m} w_j - \sum_{i=1}^{n} v_i \right) \geq p \cdot OPT \qquad \boxtimes$$

## Randomized Rounding Based on LP solution

### IP formulation

Variables $\quad y_i \in \{0,1\}$ for $i \in [n]$

$\quad z_j \in \{0,1\}$ for clauses $j \in [m]$

For clause $C_j$, we use $P_j$ to denote the set of positive literals & $N_j$ to denote negative literals

$C_j$ :

$$\bigvee_{i \in P_j} x_i \quad \vee \quad \bigvee_{i \in N_j} \bar{x}_i$$

$$\max \quad \sum_{j=1}^{m} w_j z_j$$

s.t.

$$\overbrace{\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i)} \geq z_j \qquad \text{for clause } C_j$$

$$0 \leq y_i \leq 1 \qquad i = 1, 2, \ldots, n$$

$$0 \leq z_j \leq 1 \qquad j = 1, 2, \ldots, m$$

- $Z_{LP}^* \geq Z_{IP}^* \geq OPT$

- <u>Algo:</u>

  1. Solve LP optimally to get $(y^*, z^*)$
  2. Independently for $i \in [n]$, set $x_i \leftarrow T$ w.p $y_i^*$

- <u>Analysis</u>

  Consider clause $C_j$ with $l_j$ literals

$\Pr[C_j \text{ is not satisfied}]$

AM $\geq$ GM

$l_j$ – # literals in clause $C_j$

$$= \prod_{i \in P_j} (1 - y_i^*) \cdot \prod_{i \in N_j} y_i^*$$

$$\leq \left[ \frac{1}{l_j} \left( \sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

$$= \left[ 1 - \frac{1}{l_j} \underbrace{\left( \sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right)} \right]^{l_j}$$

$$\leq \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

$f$ is concave over $[0,1]$ & $f(0) = a$

$f(1) = a+b$

then

$f(x) \leq a + bx$

$$\Pr(C_j \text{ satisfied}) \geq 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

$$\geq z_j^*\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) \qquad \left(1 - \frac{1}{l_j}\right)^{l_j} \leq e^{-1}$$

$$\geq z_j^*\left(1 - \frac{1}{e}\right)$$

$$\sum_{j \in [m]} w_j \Pr[C_j \text{ satisfied}] \geq \sum_{j \in [m]} w_j z_j^* \left(1 - \frac{1}{e}\right)$$

$\therefore$ $\left.\begin{array}{l}\text{Exp. wt. of} \\ \text{satisfied clauses}\end{array}\right\} \geq \text{OPT}\left(1 - \frac{1}{e}\right)$

$\boxtimes$

- If we set each variable to True independently w.p. $\frac{1}{2}$

[Part of reading exercise]

$$\Pr\left[C_j \text{ is satisfied}\right] \geq 1 - \frac{1}{2^{l_j}}$$

- We can improve approx. guarantee to $3/4$.

  Let $W_1$ denote the weight of soln. output by rand. LP rounding algo

  Let $W_2$ denot wt. of soln. output by $\frac{1}{2}$-prob. rounding algo.

$$W = \max\{W_1, W_2\}$$

$$\geq \frac{W_1}{2} + \frac{W_2}{2}$$

$$\geq \frac{1}{2} \cdot \sum_{j \in [m]} z_j^{*} w_j \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) + \frac{1}{2} \cdot \sum_{j \in [m]} \left(1 - \frac{1}{2^{l_j}}\right) \cdot w_j$$

$$\geq \sum_{j \in [m]} w_j z_j^{*} \underbrace{\left[\frac{1}{2}\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) + \frac{1}{2}\left(1 - \frac{1}{2^{l_j}}\right)\right]}$$

$l_j = 1$ : $\frac{1}{2}\left(1\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$

$l_j = 2$ : $\frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{4}$

$l_j^{\circ} \geq 3$

$$\frac{1}{2}\left(1-\frac{1}{e}\right)+\frac{1}{2}\cdot\frac{7}{8} \quad > \frac{3}{4} \qquad \boxtimes$$